J. Fluid Mech. (1982), vol. 124, pp. 123–137
Printed in Great Britain

Collision phenomena in free-convective flow over a sphere

By S. N. BROWN AND C. J. SIMPSON
Department of Mathematics, University College, London WC1

(Received 17 November 1981)

Two types of singularity that occur at the upper pole of a heated sphere in a fluid at rest when the Grashof number is large are discussed. The first is a property of a limit solution of the unsteady boundary-layer equations and is indicative of the fact that the boundary layer growing from the lower pole does not remain empty for all time but erupts into a plume above the sphere. The second arises from a solution of the steady boundary-layer equations and illustrates the phenomenon of an axisymmetric boundary layer converging at a point, with a velocity component parallel to the sphere that is non-zero over the major part of the boundary layer. An analysis is presented for each situation and comparison made with a numerical integration of the appropriate equations.

1. Introduction

Experimental studies by Bromham & Mayhew (1962) and by Jaluria & Gebhart (1975) have shown that at high Grashof number the flow around a heated sphere in air develops a boundary layer starting at the lower pole and evolving at the upper pole into a vertical plume of fluid that forms above the sphere. Theoretical investigations have been confined mainly to the solution of the steady boundary-layer equations, a recent study being that of Potter & Riley (1980). They present a numerical solution for the boundary layer on the sphere and discuss the eruption of the fluid at the pole. Their results compare favourably with those of earlier theories (e.g. Merk & Prins 1954; Acrivos 1960; Chiang, Ossin & Tien 1964) and with the experimental data referenced above. An interesting feature of their work is the singularity encountered at the upper pole. From their numerical evidence they predicted that at this point the skin friction and heat transfer both vanish, and the normal velocity and boundary-layer thickness become infinite.

Here we address ourselves to the same problem and show that there are two types of singularity at the upper pole: one arising from a solution of the unsteady equations after a finite time, and the second from a solution of the steady equations. We present an analysis for each type of singularity, supported in each case by a numerical integration.

For the unsteady problem we consider the situation in which the sphere in a fluid at rest is impulsively heated at time $t = 0$, and show that, if it is assumed that the meridional component of velocity parallel to the sphere vanishes at the upper pole, then such a situation cannot persist for all time, as the equations that hold in this neighbourhood develop a singularity at a non-dimensional time $t \approx 2{\cdot}912$. The singularity is in the inviscid outer part of the boundary layer, the normal velocity and maximum of the meridional velocity tending to infinity there, with the skin friction and heat transfer remaining finite and non-zero. The analytic form of the singularity is very similar to that recently discussed by Simpson & Stewartson (1982*a*) for the corresponding problem of a cylinder, and to that studied earlier by Banks & Zaturska (1979) in the neighbourhood of the equator on a spinning sphere (see also Simpson & Stewartson 1982*b*). The interpretation in the last two situations is that boundary layers with separate origins have collided after a finite time. In our problem we envisage an axisymmetric boundary layer converging at a point with non-zero velocity, and initiating an eruption into the plume above it. This is reminiscent of the boundary layer on a rotating disc in a counter-rotating fluid, discussed by Bodonyi & Stewartson (1977) (see also Stewartson, Simpson & Bodonyi 1982). For the sphere under consideration here there is good agreement between the analysis and a representative numerical solution of the equations for a Prandtl number of 0·72.

The second singularity is a property of the steady boundary-layer equations, and is that encountered at the upper pole by Potter & Riley, who integrated these equations starting at the lower pole. Their study of the equations in the neighbourhood of the upper pole supported their predictions that the limiting skin friction and heat transfer both vanish, while the normal velocity and displacement thickness become infinite. We agree with this view and present an analysis that suggests that this is an example of the collision phenomenon found by Stewartson & Simpson (1982) in their study of the flow near the entrance of a loosely coiled pipe at large Dean number. There the boundary layer on the pipe wall separates at the inner generator and a singularity develops as the axial boundary layer leaves the wall and the circumferential boundary layers collide underneath it. The singularity is novel, and not of the Goldstein type, as the separation is not caused by an adverse pressure gradient. Another example of such a singularity occurs on the lee side of a cone at incidence (see Cebeci, Brown & Stewartson 1982). For the present study of free convection from a heated sphere the analysis is similar to, and somewhat simpler than, that of the earlier studies, the position of the singularity is determined by the geometry, and the concept of an axial boundary layer forced from the wall by colliding circumferential boundary layers is replaced by that of an axisymmetric boundary layer converging on a point at which the velocity tangential to the sphere does not vanish over the major part of the boundary layer. The analysis presented here is in accord with the numerical predictions of Potter & Riley, and augments their theory of the collision. We have repeated their calculations for a Prandtl number of 0·72 as, in order to compare with the proposed theory, we required the limiting skin friction and heat transfer to an accuracy that could not be deduced from their graphical results. However, for discussion of the general properties we shall refer to these. A table of comparison shows encouraging agreement with the theory. Potter & Riley went on to discuss the eruption and subsequent development of the plume above the sphere. Such an investigation is not undertaken here and their conclusions are unaffected by the present analysis.

In the discussion we attempt to interpret the singularities in the wider context of the full unsteady boundary-layer equations and of the Navier–Stokes equations.

2. The governing equations

The dimensionless boundary-layer equations for the flow of a Boussinesq fluid over a heated sphere are, as set out by Potter & Riley (1980) but with the unsteady terms retained,

$$\frac{\partial}{\partial x}(u \sin x)+\frac{\partial}{\partial y}(v \sin x) = 0, \qquad (2.1a)$$

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=\frac{\partial^2 u}{\partial y^2}+(1-T)\sin x, \tag{2.1 b}$$

$$\frac{\partial T}{\partial t}+u\frac{\partial T}{\partial x}+v\frac{\partial T}{\partial y}=\frac{1}{\sigma}\frac{\partial^2 T}{\partial y^2}, \tag{2.1 c}$$

where t is the time, x measures angular distance from the lower pole, y measures distance normal to the sphere, u, v are velocity components in the x- and y-directions respectively, and $1-T$ is the temperature. The Prandtl number is denoted by σ and the boundary conditions for (2.1) are of zero velocity and unit temperature at the sphere, and of zero tangential velocity and temperature at infinity, so that

$$\begin{aligned} &u=v=0, \quad T=0 \quad \text{at} \quad y=0,\\ &u\to 0, \quad T\to 1 \quad \text{as} \quad y\to\infty, \end{aligned} \tag{2.2}$$

together with suitable initial conditions for the numerical integration starting at the lower pole, and a condition at $t=0$ for the case of the impulsively heated sphere.

The equations (2.1) have three independent variables. In the following sections we present numerical and analytic solutions of two limiting forms of these equations in which the number of independent variables is reduced to two. First, in §3, we take the limit $x\to\pi$ and examine the unsteady equations that result; this leads to a singularity at $t\approx 2{\cdot}912$. Secondly, in §4, we let $t\to\infty$ and show that the resulting steady equations have a singularity at the upper pole $x=\pi$.

3. The eruption of the plume at finite time

In any complete solution of (2.1) for, say, an impulsively heated sphere, with (2.2) augmented by

$$\begin{aligned} &u=v=T=0 \quad \text{at} \quad y=0,\\ &u=v=0, \quad T=1 \quad \text{for} \quad y>0 \end{aligned} \tag{3.1}$$

when $t=0$, and a condition at $x=0$ for $t>0$, it is expected that for early times, when the eruption of the plume has not taken place, $\partial v/\partial y$ in (2.1 a) will be regular at $x=\pi$. Thus u will be proportional to $\pi-x$, and since disturbances travel with the maximum speed of the fluid within the boundary layer the flow in the neighbourhood of $x=\pi$ will be independent of that over the rest of the sphere. If we take u to be proportional to $\pi-x$, and then let $x\to\pi$, any breakdown in the solution of the resulting equations will indicate that the axisymmetric boundary layer has converged at the pole and the eruption initiated. In (2.1) we therefore write

$$u=(\pi-x)\bar{u} \tag{3.2}$$

and let $x\to\pi$. This leads to

$$2\bar{u}=\frac{\partial v}{\partial y}, \quad \frac{\partial \bar{u}}{\partial t}-\bar{u}^2+v\frac{\partial \bar{u}}{\partial y}=1-T+\frac{\partial^2\bar{u}}{\partial y^2}, \tag{3.3 a, b}$$

$$\frac{\partial T}{\partial t}+v\frac{\partial T}{\partial y}=\frac{1}{\sigma}\frac{\partial^2 T}{\partial y^2}, \tag{3.3 c}$$

which are to be solved subject to (2.2), (3.1). The presence of the factor 2 in (3.3a) is the only difference between these equations and those for the cylinder considered by Simpson & Stewartson (1982a). The properties of the numerical solution, which we carried out here by the same method for $\sigma=0{\cdot}72$ as employed there for unit Prandtl number, are similar to those of the solution they obtained. The solution comes

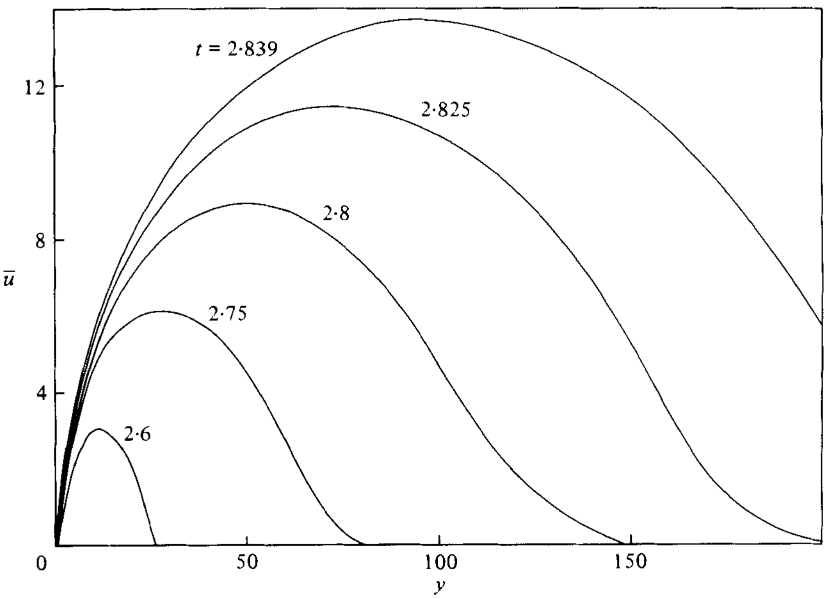

FIGURE 1. Profiles of $\bar{u}$ for various values of t, illustrating the increase of the maximum value $\bar{u}_{\max}$.

to an end at a finite time, the value of which we estimate below, the non-dimensional skin friction and heat transfer remain finite and non-zero, but an essentially inviscid singularity develops in the middle of the boundary layer with $\bar{u}$ becoming infinite there. Profiles of $\bar{u}$ exhibiting this property are presented in figure 1.

The numerical solution indicated that the inverse $\bar{u}_{\max}^{-1}$ of the maximum value of $\bar{u}$, was decreasing linearly as the computation was coming to an end. This implies a balance of the terms $\partial\bar{u}/\partial t$ and $\bar{u}^2$ in (3.3b), and it seemed likely that the singularity was of the same form as that for the cylinder. By analogy with the solution there, we therefore set

$$\tau = t_s - t, \tag{3.4}$$

where t_s is the unknown time at which breakdown occurs, and seek a solution of (3.3) of the form

$$\bar{u} = \frac{1}{\tau}\sum_{n=0}^{\infty}\tau^{\frac{1}{2}n}u_n(z), \quad v = \frac{1}{\gamma\tau^{\frac{5}{2}}}\sum_{n=0}^{\infty}\tau^{\frac{1}{2}n}v_n(z), \quad T = \sum_{n=0}^{\infty}\tau^{\frac{1}{2}n}T_n(z) \tag{3.5}$$

when $\tau \ll 1$, and

$$z = 2\gamma\tau^{\frac{3}{2}}y, \tag{3.6}$$

γ being an unknown constant. It emerges that in order to obtain a satisfactory match with a solution valid in the neighbourhood of the wall the series (3.5) must also contain terms in $\tau^{\frac{1}{2}n}(\log\tau)^m$. The first, with $m = 1$, occurs when $n = 3$.

From (3.3a, b) we first obtain the equations for the functions u_0, v_0. The term $1-T$ is of relative order τ^2 and the viscous term of relative order τ^4, so the right-hand side of (3.3b) is negligible until $n = 4$. *For* u_0, v_0 we have

$$u_0 = v_0', \quad u_0 - \tfrac{3}{2}zu_0' - u_0^2 + 2v_0u_0' = 0, \tag{3.7a, b}$$

which yield, on differentiation of (3.7b) and elimination of v_0,

$$u_0''(u_0 - u_0^2) = -\tfrac{1}{2}u_0'^2. \tag{3.8}$$

the integration of which is elementary. If we set $u_0' = H(z)$, incorporate the constant of integration into γ, and insist that H does not become exponentially large with z, it follows that the solution of (3.8) is

$$u_0 = (1+H^2)^{-1}, \tag{3.9}$$

where

$$\frac{dH}{dz} = -\tfrac{1}{2}(1+H^2)^2. \tag{3.10}$$

From (3.10) we obtain the relation between z and H, with $z \to 0$ as $H \to \infty$, so that $u_0 \to 0$ as $z \to 0$, as we anticipate that $\bar{u}$ will be a regular function of τ in the wall layer with which this solution must match: this implies that

$$z = \operatorname{arccot} H - \frac{H}{1+H^2}. \tag{3.11}$$

Then from (3.7*b*) the corresponding solution for v_0 is

$$v_0 = \tfrac{3}{4}z - \frac{H}{2(1+H^2)^2}. \tag{3.12}$$

From (3.3*c*) we now have T to $O(\tau^{\frac{1}{2}})$. The result is

$$T = a_0 + a_1 \tau^{\frac{1}{2}} H + O(\tau), \tag{3.13}$$

where a_0 and a_1 are arbitrary constants.

This leading-order solution requires some comment. In the case of the impulsively heated cylinder, Simpson & Stewartson found the corresponding solution by writing the inviscid equations in terms of a suitably defined stream function. The same technique may be employed here, the advantage of it being that the variable z in (3.6) and the forms (3.5) for $\bar{u}$, v, T emerge naturally. However it is cumbersome for the calculation of the higher-order terms, so in the interest of brevity we have omitted it here, and have assumed (3.5), (3.6) *ab initio*. The significance of H and its relationship with z is as follows. The numerical work indicates that $\bar{u}$ has a maximum, the value of which tends to infinity as $\tau \to 0$ in the outer part of the boundary layer now under discussion. At this value of z, $u_0' = H = 0$, and H increases as z decreases, and decreases as z increases. As $H \to \infty$, $z \to 0$ with, from (3.11), $H \approx (\frac{3}{2}z)^{-\frac{1}{3}}$ and as $H \to -\infty$, $z \to \pi$ with $H \approx -(\frac{3}{2}(\pi - z))^{-\frac{1}{3}}$. Thus, in terms of z, this outer inviscid singular layer is confined to $(0, \pi)$. Below it there is a viscous boundary layer and above the viscous terms are again important. The constant of integration in (3.11) was chosen so that $z \to 0$ as $H \to \infty$ and this layer may be matched directly to the wall layer. The value of $\bar{u}_{\max}$ from the numerical work provides an immediate check on the theory, since it follows from (3.3*b*) that, when $\partial\bar{u}/\partial y = 0$,

$$\frac{\partial}{\partial t}(\bar{u}_{\max}) = \bar{u}^2_{\max}, \tag{3.14}$$

with a relative error of order τ^2. Thus

$$\bar{u}_{\max} = \frac{1}{\tau} + O(\tau) \tag{3.15}$$

the leading term of which is given by (3.9) with $H = 0$. In table 1 we tabulate $\bar{u}_{\max}$, its inverse, and the value of v at the edge of the boundary layer, all as given by the numerical work. Then by consideration of $\bar{u}_{\max} - \tau^{-1}$ for various t_s we infer that

$$t_s \approx 2{\cdot}912. \tag{3.16}$$

t	$\bar{u}_{max}$	$1/\bar{u}_{max}$	v_∞
0	0		0
1·0	0·27187	3·6782	1·2120
2·0	0·93480	1·0698	9·6836
2·5	2·3433	0·42676	65·913
2·6	3·1405	0·31842	128·41
2·7	4·6727	0·21401	328·20
2·8	8·9058	0·11229	1575·9
2·805	9·3242	0·10725	1763·8
2·810	9·7835	0·10221	1985·0
2·815	10·290	0·09718	2247·6
2·820	10·851	0·09215	2561·9
2·825	11·477	0·08713	2942·0
2·830	12·180	0·08210	3406·6
2·835	12·973	0·07708	3981·7

TABLE 1. The maximum value of $\bar{u}$ in the boundary layer, the inverse of this value, and the value of v at the edge of the boundary layer

The arbitrary constant γ will also be determined by the numerical work, but before so doing we calculate u_1, v_1 and discuss the solutions in the viscous regions near $z = 0$ and beyond $z = \pi$, as these provide restrictions on the inviscid solution. The equations for u_1, v_1 are

$$u_1 = v_1', \quad \frac{H}{(1+H^2)^2}v_1'' - \left(\frac{1}{2} - \frac{2}{1+H^2}\right)v_1' - 2Hv_1 = 0, \tag{3.17}$$

so that

$$u_1 = \frac{2A_1H}{1+H^2} - 2B_1H, \quad v_1 = \frac{A_1}{(1+H^2)^2} + B_1\left(1 - \frac{2}{1+H^2}\right), \tag{3.18}$$

where A_1, B_1 are arbitrary constants. We anticipate that in the inner layer the solution of (3.3) will be of the form

$$\bar{u} = \frac{1}{2}\sum_{n=0}^{\infty} \tau^n \bar{v}_n'(y), \quad v = \sum_{n=0}^{\infty} \tau^n \bar{v}_n(y), \quad T = \sum_{n=0}^{\infty} \tau^n \bar{T}_n(y), \tag{3.19}$$

with only integral powers of τ appearing as the numerical work indicates that the solution is regular there. Here $\bar{v}_n(0) = \bar{v}_n'(0) = \bar{T}_n(0) = 0$ to satisfy the boundary conditions on the sphere; $\bar{v}_0$, $\bar{T}_0$ are otherwise arbitrary and will depend on the previous history of the boundary layer, since $\bar{u}$, v, T in (3.19) are assumed to be regular functions of τ, except that

$$\bar{v}_0'(y) \approx 2(3\gamma y)^{\frac{2}{3}}, \quad \bar{T}_0 \approx a_0 \quad \text{as} \quad y \to \infty, \tag{3.20}$$

in order that the leading terms of $\bar{u}$, T may match, as $y \to \infty$, with those given by (3.10), (3.7) as $z \to 0$ ($H \to \infty$). Subsequent $\bar{v}_n$, $\bar{T}_n$ may then be calculated successively. For example

$$\bar{v}_1'(y) = -\tfrac{1}{2}\bar{v}_0'^2 + \bar{v}_0\bar{v}_0'' - 2(1-\bar{T}_0) - \bar{v}_0''', \tag{3.21}$$

$$\bar{T}_1(y) = \bar{v}_0\bar{T}_0 - \frac{1}{\sigma}\bar{T}_0''. \tag{3.22}$$

Consider now the term $-2B_1H$ in u_1 in (3.18). As $H \to \infty$ this will contribute a term $O(\tau^{-1})$ to the inner solution, which is unacceptable. Thus $B_1 = 0$.

In the outer viscous region beyond $z = \pi$ we expect the solution to be of the form

$$\bar{u} = \frac{1}{2}\sum_{n=0}^{\infty} \tau^n \bar{V}_n'(Y), \quad v = h'(\tau) + \sum_{n=0}^{\infty} \tau^n \bar{V}_n(Y), \quad \bar{T} = \sum_{n=0}^{\infty} \tau^n \bar{T}_n(Y), \tag{3.23}$$

where

$$Y = y + h(\tau), \tag{3.24}$$

and $h'(\tau)$ is the value of v given by the solution in the centre inviscid region as $z \to \pi$. Thus, from (3.12), we have the leading contribution

$$h'(\tau) \approx \frac{3\pi}{4\gamma\tau^{\frac{5}{2}}}, \tag{3.25}$$

and this is also the leading contribution to the value v_∞ of v as $y \to \infty$. The results for v_∞ from the numerical work are also displayed in table 1, from which we estimate 0·3661 as the best-fit value for γ. In (3.23) the functions $\bar{V}_0$, $\bar{T}_0$ are arbitrary, except that

$$\bar{V}_0' \to 0, \quad \bar{T}_0 \to 1 \quad \text{as} \quad Y \to \infty, \tag{3.26}$$

to satisfy the conditions as $y \to \infty$, and

$$\bar{V}_0' \approx 2(-3\gamma Y)^{\frac{2}{3}}, \quad \bar{T}_0 \approx a_0 \quad \text{as} \quad Y \to -\infty, \tag{3.27}$$

so that $\bar{u}$, T in (3.23) may, to leading order, match with (3.10), (3.7) as $H \to -\infty$ ($z \to \pi$). Successive $\bar{V}_n$, $\bar{T}_n$ may be calculated without difficulty.

To find the subsequent singular terms in v_∞ as $\tau \to 0$ it is necessary to continue the expansions (3.5) beyond the terms v_0, v_1 already found. For u_2, v_2 we obtain

$$u_2 = 2A_1^2\left(\frac{3}{2} - \frac{2}{1+H^2}\right) + A_2\left(4 - \frac{4}{1+H^2} + 2zH\right), \tag{3.28}$$

$$v_2 = \frac{2A_1^2 H}{(1+H^2)^2} + A_2\left(\frac{3H}{(1+H^2)^2} - \tfrac{1}{2}z + \frac{2z}{1+H^2}\right), \tag{3.29}$$

where A_2 is a second arbitrary constant and the coefficient B_2 of the other complementary function has been set equal to zero for the same reason as was B_1 in (3.18). From v_2 we obtain a contribution $-A_2\pi/2\gamma\tau^{\frac{3}{2}}$ to v_∞, which is also a contribution to $h'(\tau)$ in (3.25). The term $2A_2zH$ as $z \to \pi$ in (3.28) is to be considered in conjunction with the term H^{-2} in u_0 for $|H| \gg 1$, and is required in order that, as $z \to \pi$, $\bar{u}$ shall consist of a series of integral powers of τ with coefficients that are functions of Y. It must be remembered that we have adjusted the definition of Y at this stage because of the correction to $h'(\tau)$. In fact, at the u_3, v_3 stage of the expansion insistence that the solution remains of the required form suggests that A_1 should be zero, so, as it gives no contribution to $\bar{u}_{\max}$ or v_∞, we set $A_1 = 0$ and obtain

$$u_3 = 2A_3H\left(\log(1+H^2) + \frac{1}{1+H^2} - 1\right) + B_3H, \tag{3.30}$$

$$v_3 = A_3\left(\frac{2\log(1+H^2)}{1+H^2} + \frac{1}{(1+H^2)^2} - 1\right) + \frac{B_3}{1+H^2}, \tag{3.31}$$

where A_3, B_3 are arbitrary constants. The term proportional to $H\log|H|$ in u_3 for $|H| \gg 1$ will force a logarithmic term on the wall boundary layer unless a term $O(\tau^{\frac{1}{2}}\log\tau)$ in $\bar{u}$ and $O(\tau^{-1}\log\tau)$ in v are included in (3.5). The term to add to $\bar{u}$ is $(\tau^{\frac{1}{2}}\log\tau)2A_3H$ and to v is $(\tau^{-1}\log\tau)2A_3/(1+H^2)$. If these terms are taken in conjunction with (3.30), (3.31) then the wall boundary layer and the viscous layer beyond $z = \pi$ have no logarithmic terms at this stage. The term in $\tau^{-1}\log\tau$ gives a contribution to the equation for v_3, the solution of which is now

$$v_3 = A_3\left(\frac{2\log(1+H^2)}{1+H^2}+\frac{1}{(1+H^2)^2}\right)+\frac{B_3}{1+H^2} \tag{3.32}$$

instead of (3.31), so there is no term $O(\tau^{-1})$ in v_∞.

It may be shown that v_∞ has a term $O(\tau^{-\frac{1}{2}})$ by considering the equation for v_4, which has the acceptable complementary function

$$v_4 = A_4\left\{\frac{25H}{6(1+H^2)}+\frac{7H(5+H^2)}{6(1+H^2)^2}+z\left(1+\frac{4}{1+H^2}\right)\right\} \tag{3.33}$$

for arbitrary constant A_4. Thus altogether we have

$$v_\infty = h'(\tau) = \frac{\pi}{\gamma\tau^{\frac{5}{2}}}(\tfrac{3}{4}-\tfrac{1}{2}A_2\tau+A_4\tau^2)+o(\tau^{-\frac{1}{2}}), \tag{3.34}$$

where A_2, A_4, as well as γ, can be estimated from the numerical work. The fit is satisfactory with $\gamma = 0{\cdot}3661$ and $A_2 = -0{\cdot}38$, $A_4 = 0{\cdot}22$. Further constants, for example a_0, a_1, can be estimated if required.

This completes the discussion of the singularity that occurs at the upper pole at finite time. In §4 we analyse a singularity of a solution of the steady equations that occurs as $x \to \pi$.

4. The singularity of the steady equations as $x \to \pi$

In this section we address ourselves to the problem considered by Potter & Riley (1980). They integrated the steady form of (2.1) from $x = 0$ and showed that as $x \to \pi$ the solution becomes singular, the skin friction and heat transfer tend to zero and the normal velocity and boundary-layer thickness become infinite. We extend their analysis of the singularity, showing that it may be regarded as an example of the colliding boundary-layer phenomenon first resolved by Stewartson & Simpson (1982) in the case of entry flow in a curved pipe. Here the axisymmetric boundary layer reaches the upper pole with non-zero tangential velocity over the main portion of the layer and the plume erupts. As in the situation of the curved pipe there are two regions to discuss, one near the wall and a second outer inviscid region below the original boundary layer that has been forced away from the wall. Unlike the unsteady problem of §3, where the singularity is essentially inviscid, with the viscous regions remaining almost regular as the singularity in time is approached, here it is the viscous regions that are most affected by the singularity.

To study the neighbourhood of $x = \pi$ we write

$$X = \pi - x, \tag{4.1}$$

where $X \ll 1$. We first consider the region $y \ll 1$ and, since figure 3 of Potter & Riley indicates that near $x = \pi$ the temperature is almost constant near the wall, we replace the steady form of (2.1) by

$$X\frac{\partial u}{\partial X}+u-X\frac{\partial v}{\partial y} = 0, \tag{4.2a}$$

$$-u\frac{\partial u}{\partial X}+v\frac{\partial u}{\partial y} = \frac{\partial^2 u}{\partial y^2}+X, \tag{4.2b}$$

$$-u\frac{\partial T}{\partial X}+v\frac{\partial T}{\partial y}=\frac{1}{\sigma}\frac{\partial^2 T}{\partial y^2}, \tag{4.2c}$$

with a relative error that turns out to be $O(X)$ modified by an appropriate power of $\log X^{-1}$.

When $y \ll 1$ the required variable is of the same form as that employed by Stewartson & Simpson (1982) for the curved-pipe problem, and we write

$$\eta = y\mu^{\frac{1}{3}}, \tag{4.3}$$

where $\mu(\gg 1)$ is a function of X to be found. We then express u, v, T in (4.2) in the forms

$$u = X\mu^{\frac{2}{3}}U(\mu,\eta), \quad v=\mu^{\frac{1}{3}}V(\mu,\eta), \quad T = pX\mu^{\alpha}S(\mu,\eta), \tag{4.4}$$

where α, p are constants to be found. The value of α will be determined by the requirement of self-consistency of the inner solution, but p will remain arbitrary. We suspect that, like the other two constants that will emerge from the inner solution, it is determined in part by the previous history of the boundary layer. The functions U, V, S are then expanded as

$$U = \eta + \sum_{n=1}^{\infty} \mu^{-\frac{4}{3}n} U_n(\eta), \quad V = \eta^2 + \sum_{n=1}^{\infty} \mu^{-4n/3} V_n(\eta),$$
$$S = \eta + \sum_{n=1}^{\infty} \mu^{-4n/3} S_n(\eta), \tag{4.5}$$

where we require $U_n(0) = V_n(0) = S_n(0) = 0$, and that U_n, V_n, S_n are not exponentially large as $\eta \rightarrow \infty$.

The equations for U, V, S are, from (4.3), (4.4),

$$2U-\frac{\partial V}{\partial \eta}-\tfrac{2}{3}\delta U-\tfrac{1}{3}\delta\eta\frac{\partial U}{\partial \eta}-\delta\mu\frac{\partial U}{\partial \mu}=0, \tag{4.6a}$$

$$\frac{\partial^2 U}{\partial \eta^2}-V\frac{\partial U}{\partial \eta}+(1-\tfrac{2}{3}\delta)U^2-\tfrac{1}{3}\delta\eta U\frac{\partial U}{\partial \eta}-\delta\mu U\frac{\partial U}{\partial \mu}+\mu^{-\frac{4}{3}}=0, \tag{4.6b}$$

$$\frac{1}{\sigma}\frac{\partial^2 S}{\partial \eta^2}-V\frac{\partial S}{\partial \eta}+(1-\alpha\delta)US-\tfrac{1}{3}\delta\eta U\frac{\partial S}{\partial \eta}-\delta\mu U\frac{\partial S}{\partial \mu}=0, \tag{4.6c}$$

where we have defined

$$\delta = -\frac{X\mu'}{\mu} \tag{4.7}$$

and shall find that $\delta = O(\mu^{-\frac{4}{3}})$.

As (4.6a, b) are independent of S the analysis is simpler than that of Stewartson & Simpson. The equations for U_1, V_1 are

$$2U_1-V_1'-\delta\mu^{\frac{4}{3}}\eta=0, \tag{4.8a}$$
$$U_1''-V_1-\eta^2U_1'+2\eta U_1-\delta\eta^2\mu^{\frac{4}{3}}+1=0, \tag{4.8b}$$

and on eliminating V_1 we deduce that

$$U_1''(\eta) = -\delta\mu^{\frac{4}{3}}e^{\frac{1}{3}\eta^3}\int_{\eta}^{\infty}\eta_1 e^{-\frac{1}{3}\eta_1^3}d\eta_1 \tag{4.9}$$

and, since $U_1''(0) = -1$,

$$\delta\mu^{\frac{4}{3}} = \frac{3^{\frac{1}{3}}}{(-\frac{1}{3})!} = 1{\cdot}0650 = \alpha_1, \text{ say.} \tag{4.10}$$

Thus the relationship between μ and X is

$$\mu^{\frac{4}{3}} = \tfrac{4}{3}\alpha_1 \log \frac{\beta_1}{X}, \tag{4.11}$$

where β_1 is an arbitrary constant.

From (4.6*c*) the equation for S_1 is

$$\frac{1}{\sigma} S_1'' - \eta^2 S_1' + \eta S_1 = V_1 - \eta U_1 + \alpha_1(\alpha + \tfrac{1}{3})\,\eta^2, \tag{4.12}$$

and since this has a complementary function, namely η, that satisfies both boundary conditions it will have an acceptable solution only if α satisfies a certain condition, which is easily found to be

$$\int_0^\infty \eta\, e^{-\sigma\eta^3/3}(V_1 - \eta U_1 + \alpha_1(\alpha + \tfrac{1}{3})\,\eta^2)\,d\eta = 0. \tag{4.13}$$

On use of (4.9) to substitute for U_1, and (4.8*a*) for V_1, this integral may be evaluated to give α as a function of σ in the form

$$\tfrac{2}{3} - \alpha = \frac{b_1}{\alpha_1^2}\,\sigma^{\frac{2}{3}} - \tfrac{1}{3}(1-\sigma)^{\frac{1}{3}}\,\sigma^{\frac{1}{3}} B(\sigma) + 1 - \sigma, \tag{4.14}$$

where

$$\frac{1}{b_1} = \int_0^\infty e^{-\eta^3/3}\,d\eta = 3^{\frac{1}{3}}(\tfrac{1}{3})! = 1{\cdot}2879, \tag{4.15}$$

and $B(\sigma)$ is the incomplete beta function:

$$B(\sigma) = \int_0^{1-\sigma} x^{-\frac{1}{3}}(1-x)^{-\frac{1}{3}}\,dx. \tag{4.16}$$

If $\sigma = 1$, $\alpha = -0{\cdot}01791$; if $\sigma = 0{\cdot}72$, $\alpha = -0{\cdot}03245$ and if $\sigma = 0$, $\alpha = -\frac{1}{3}$. As in the problem of the curved pipe, the equations for U_n, V_n, S_n have a complementary function, which in this case is

$$U_n = \gamma_n \eta, \quad V_n = \gamma_n \eta^2, \quad S_n = c_n \eta \quad (n \geqslant 1), \tag{4.17}$$

where γ_n, c_n are constants, and, as there, each γ_n is determined by a compatibility condition on the equation for U_{n+1}, except for γ_1, which is indeterminate, and the compatibility requirement may then be satisfied by the addition of a term $O(\mu^{-\frac{8}{3}})$ to δ so that (4.10) becomes

$$\delta = \alpha_1 \mu^{-\frac{4}{3}} + \alpha_1^2 \alpha_2 \mu^{-\frac{8}{3}}, \tag{4.18}$$

and (4.11) is replaced by

$$\mu^{\frac{4}{3}} - \alpha_1\alpha_2 \log(\mu^{\frac{4}{3}} + \alpha_1\alpha_2) = \tfrac{4}{3}\alpha_1 \log\frac{\beta_1}{X}. \tag{4.19}$$

Here we have calculated

$$\alpha_2 = \tfrac{1}{9} - \tfrac{1}{6}\log 3 - \tfrac{1}{18}\sqrt{\tfrac{1}{3}}\,\pi = -0{\cdot}1728. \tag{4.20}$$

Each c_n is determined by the requirement that the equation for S_{n+1} has an acceptable solution, and in all the expansion contains the three arbitrary constants β_1, γ_1 and p, which depend on the flow conditions in $X > 0$. We now compare this theory with a numerical integration of (2.1).

We integrated the steady form of (2.1) from $x = 0$ with $\sigma = 0{\cdot}72$ as did Potter & Riley. The general form of the solution was evident from their work, but we wished

x	ζ	s
3·0	0·08703	0·3243
3·01	0·08104	0·3247
3·02	0·07503	0·3250
3·03	0·06900	0·3254
3·04	0·06294	0·3257
3·05	0·05686	0·3261
3·06	0·05075	0·3264
3·07	0·04462	0·3268

TABLE 2. Values of ζ and s, as defined in (4.21) and (4.22), calculated from the numerical integration

to make a qualitative comparison with the detailed structure outlined above. We make two comparisons, one on the skin friction and the other on the heat transfer. Using (4.4), (4.5), we first identify μ with a non-dimensional skin friction $\tau = X^{-1}(\partial u/\partial y)_{y=0}$, since $\mu/\tau = 1+O(\mu^{-\frac{4}{3}})$, and tabulate

$$\zeta = (\tau^{\frac{4}{3}}+\alpha_1\alpha_2)^{3\alpha_2/4}\exp\left(-\frac{3}{4\alpha_1}\tau^{\frac{4}{3}}\right) \tag{4.21}$$

against x, which from (4.19) should be close to $(\pi-x)/\beta_1$ as $x\to\pi$. A similar test was applied by Stewartson & Simpson. The results are shown in table 2, which would indicate 3·1416 as the position of the singularity with $\beta_1 \approx 1{\cdot}58$. This is a much more severe test of the theory than a check that the skin friction vanishes at $x=\pi$, since we have already removed the factor $\pi - x$, and in view of the order of magnitude of the errors in the formula (4.21), and the distance of the end of the range of integration at $x = 3{\cdot}07$ from the pole $x=\pi$ it is felt that the comparison is fairly convincing.

Also in table 2 we tabulate

$$s = X^{-1}\tau^{-\alpha-\frac{1}{3}}(\partial T/\partial y)_{y=0}, \tag{4.22}$$

which should tend to the constant p as $x\to\pi$. We note that s is almost constant over a wide range of x and infer that $p\approx 0{\cdot}328$.

We now proceed to examine the outer structure near the singularity. The main purpose of this investigation is to ensure that it can be matched to the inner structure.

In the region away from the immediate neighbourhood of $y=0$ we assume that the solution is inviscid, and then, on defining a stream function ψ such that

$$u\sin x = \frac{\partial\psi}{\partial y},\quad v\sin x = -\frac{\partial\psi}{\partial x}, \tag{4.23}$$

we may write the steady solution of (2.1) in terms of two arbitrary functions of ψ. First, from (2.1*c*),

$$T = F(\psi), \tag{4.24}$$

and then, from (2.1*b*),

$$\tfrac{1}{2}u^2 = -(1-T)\cos x + f(\psi). \tag{4.25}$$

This means that, near $x=\pi$,

$$\frac{\partial\psi}{\partial y} \approx uX \approx 2^{\frac{1}{2}}X\{f(\psi)+1-F(\psi)\}^{\frac{1}{2}}, \tag{4.26}$$

and hence that ψ is of the form

$$\psi = \bar{H}(Xy - g(X)) \tag{4.27}$$

for arbitrary functions $\bar{H}$, g. This solution is consistent with that of Potter & Riley, who in their figures 2 and 3 show that the velocity and temperature become functions of Xy as $X \to 0$. We shall argue below that $g(0)$ is finite. The corresponding velocity component v takes the form

$$v = \frac{y - g'(X)}{X} \bar{H}'(Xy - g(X)), \tag{4.28}$$

which is $O(X^{-2})$ in a region in which $y = O(X^{-1})$, as predicted by Potter & Riley in their figure 5. If we define

$$Y_1 = Xy - g(X), \tag{4.29}$$

the solution when $Y_1 = O(1)$ takes on the form

$$\psi = H_0(Y_1) + CXH_0'(Y_1) + O(X^2), \tag{4.30}$$

$$T = T_0(Y_1) + CXT_0'(Y_1) + O(X^2), \tag{4.31}$$

where C is an arbitrary constant, the viscous terms being important at $O(X^3)$. Here H_0', $T_0 \to 0$ as $Y_1 \to -\infty$. Evidence in support of the interpretation of the phenomenon as a collision is provided by the fact that $u = H_0'(Y_1)$ when $Y_1 = O(1)$ and is not proportional to X. Thus away from the immediate neighbourhood of the sphere the limiting tangential velocity does not vanish.

The inviscid solution given by (4.24), (4.25) must match, as $\psi \to 0$, with the solution (4.5) as $\eta \to \infty$. We shall gain no further information from this match except that the theory is consistent. In the curved-pipe problem the asymptotic form of the quantity corresponding to $g(X)$ was determined at this stage, but here it emerges that the appropriate integral is finite and so its value can be obtained only by knowledge of the integrand over the whole range, which we do not have.

Now it follows from (4.5), (4.9) that, when $\eta \gg 1$,

$$u = X\mu^{\frac{2}{3}}\{\eta + \mu^{-\frac{4}{3}}(-\alpha_1 \eta \log \eta + \Gamma_1 \eta + \dots) + O(\mu^{-\frac{8}{3}})\}, \tag{4.32}$$

where Γ_1 is related to the unknown constant γ_1, so that, as $\mu \to \infty$ with y fixed,

$$u = Xy\mu\{1 - \tfrac{1}{3}\alpha_1 \mu^{-\frac{4}{3}} \log \mu + O(\mu^{-\frac{4}{3}})\}. \tag{4.33}$$

Thus, since $\partial\psi/\partial y = Xu$ we have

$$\psi = \tfrac{1}{2}X^2y^2\mu\{1 - \tfrac{1}{3}\alpha_1 \mu^{-\frac{4}{3}} \log \mu + O(\mu^{-\frac{4}{3}})\}. \tag{4.34}$$

But from (4.19) we infer that

$$\mu^{\frac{4}{3}} = \Psi + \alpha_1(\alpha_2 + \tfrac{1}{2}) \log \Psi + O(1) \tag{4.35}$$

when $\Psi \gg 1$, where

$$\Psi = \tfrac{4}{3}\alpha_1 \log \frac{\beta_1}{(2\psi)^{\frac{1}{2}}}, \tag{4.36}$$

and hence from (4.26) that

$$f(\psi) + 1 - F(\psi) = \psi\Psi^{\frac{3}{4}}\left\{1 + \alpha_1(\tfrac{3}{4}\alpha_2 + \tfrac{1}{8}) \frac{\log \Psi}{\Psi} + O\left(\frac{1}{\Psi}\right)\right\} \tag{4.37}$$

when $\psi \ll 1$.

In a similar way it may be shown that

$$F(\psi) = p(2\psi)^{\frac{1}{2}}\Psi^{\frac{1}{4}(3\alpha-\frac{1}{2})}\left\{1+\tfrac{1}{4}\alpha_1(\tfrac{17}{12}+\tfrac{1}{2}\alpha+\alpha_2(3\alpha-\tfrac{1}{2}))\frac{\log\Psi}{\Psi}+O(\Psi^{-1})\right\}. \tag{4.38}$$

Since $\{f(\psi)+1-F(\psi)\}^{\frac{1}{2}} = O(\psi^{\frac{1}{2}}\Psi^{\frac{3}{8}})$ as $\psi \rightarrow 0$ the formula $Xu = \partial\psi/\partial y$ gives

$$Xy = \int^{\psi}\frac{d\psi}{u}, \tag{4.39}$$

and the integral converges as $\psi \rightarrow 0$. Thus $g(0)$ is finite and its value cannot be found by study of the local properties only of the singularity. In the inviscid region, therefore, the leading-order contribution to u and v may be written as

$$u = \overline{H}_0(Xy), \quad v = y\overline{H}_0(Xy)/X \tag{4.40}$$

for an arbitrary function $\overline{H}_0$. This clearly cannot be valid right to the edge of the boundary layer, since $v < 0$ there, while from (4.40) v has the sign of u and is large and positive for small X. For some Xy the viscous terms of (2.1) must be non-negligible again, and the outer viscous solution is either of similarity form, or it depends on the previous history of the boundary layer. We have been unable to find an appropriate similarity solution and therefore favour the latter. Indeed, if Mangler's (1945) transformation is applied to the steady form of (2.1), although they cannot be reduced entirely to two-dimensional form, the appropriate variable in the direction normal to the sphere is found to be Xy, which would indicate that the solution in the viscous layer is also of the form (4.30), (4.31), with $\overline{H}_0$ in (4.40) exponentially small as $Xy \rightarrow \infty$. Subsequent terms in the expansion would allow for a change of sign of v. However, Mangler's transformation is itself singular as $X \rightarrow 0$ and can only be used to suggest that Xy is the appropriate variable in both the inviscid and outer viscous regions. We were unable to make a conclusive deduction from the numerical work, as in this region the quantities were all very small, typically $O(10^{-3})$.

5. Discussion

It is of interest to try to interpret the singularities discussed here in the wider context of the solution of the full unsteady boundary-layer equations for the complete flow around the sphere, or even in that of the Navier–Stokes equations. For a solution of the unsteady boundary-layer equations the conditions required are a specification of the whole flow and temperature fields at $t = 0$, and boundary conditions on the sphere and at infinity at all times together with an initial condition at the lower pole $x = 0$. For sufficiently small time, $\partial v/\partial y$ will be regular at $x = \pi$, the meridional velocity u will vanish there and the flow near the upper pole may be discussed independently of that over the rest of the sphere. This continues until the equations exhibit the singularity at finite time that is the subject of §3. We interpret the singularity as heralding the collision of the axisymmetric boundary layer with itself, and it would be of interest to see if the time coincided with that at which a solution of the full unsteady equations predicted a boundary layer that was no longer empty at the upper pole. An exactly analogous situation occurs in the case of the cylinder discussed by Simpson & Stewartson (1982*a*), except that there two boundary layers of disjoint origin are colliding at the upper generator.

One can make conjectures about the fate of the solution of the full unsteady boundary-layer equations after the time at which the axisymmetric boundary layer has converged on itself, or, in the case of the cylinder, at which the tangential velocity is no longer zero at the upper generator. In the latter geometry the situation seems fairly clear. Each boundary layer is unaware of the presence of the other, so it would be expected that the solution would continue free of singularities as long as the flow does not separate at a finite time. If it did, a viscous singularity of the Moore–Rott–Sears type (Sears & Telionis 1975) would most likely be encountered. However, the studies of Ingham (1978) indicate that this is not so, but that at large time the solution approaches that of Merkin (1976), which does not separate until $x > \pi$. Thus for $0 < x < \pi$ we would expect a solution regular for all t. For the sphere, however, the picture is not so clear because of the singularity of the steady equations at $x = \pi$ and because of the fact that the boundary layer is immediately aware of the collision and eruption that occurs when it is no longer empty at the upper pole. One possibility is that the singularity that we have found of the unsteady equations at the upper pole is in fact spurious and the solution of the full unsteady equations will either terminate in a Moore–Rott–Sears singularity or will not exhibit any non-zero tangential velocity at the upper pole at any finite time. However, this is not in accord with experimental observations, which confirm a definite eruption of the plume. It is likely that this will occur at a time that terminates the validity of the boundary-layer equations in the neighbourhood of $x = \pi$. From then on, since the meridional velocity is no longer zero there, it is necessary to match with a solution of the full Navier–Stokes equations valid in that region. The boundary-layer solution may then be regarded as an outer solution. We anticipate that as $t \to \infty$ this outer solution tends, as $x \to \pi$, to the singular solution of the steady equations discussed in §4, though further large-scale numerical solutions are required before this conjecture can be confirmed.

We are grateful to Professor K. Stewartson for his encouragement with this problem.

REFERENCES

Acrivos, A. 1960 A theoretical analysis of laminar natural convection heat transfer to non-Newtonian fluids. *A.I.Ch.E. J.* **6**, 584–590.

Banks, W. H. H. & Zaturska, M. B. 1979 The collision of unsteady laminar boundary layers. *J. Engng Math.* **13**, 193–212.

Bodonyi, R. J. & Stewartson, K. 1977 The unsteady laminar boundary layer on a rotating disc in a counter-rotating fluid. *J. Fluid Mech.* **79**, 669–688.

Bromham, R. J. & Mayhew, Y. R. 1962 Free convection from a sphere in air. *Int. J. Heat Mass Transfer* **5**, 83–84.

Cebeci, T., Brown, S. N. & Stewartson, K. 1982 Non-similar boundary layers on the lee side of cones at incidence. To be submitted to *Phys. Fluids*.

Chiang, T., Ossin, A. & Tien, C. L. 1964 Laminar free convection from a sphere. *Trans. A.S.M.E.* C: *J. Heat Transfer* **86**, 537–542.

Ingham, D. B. 1978 Free convection boundary layer on an isothermal horizontal cylinder. *Z. angew. Math. Phys.* **29**, 871–883.

Jaluria, Y. & Gebhart, B. 1975 On the buoyancy-induced flow arising from a heated hemisphere. *Int. J. Heat Mass Transfer* **18**, 415–431.

Mangler, W. 1945 Boundary layers on bodies of revolution in symmetrical flow. *Ber. aerodyn. VersAnst. Göttingen* 45/A/17.

Merk, H. J. & Prins, J. A. 1954 Thermal convection in laminar boundary layers. III. *Appl. Sci. Res.* A **4**, 207–221.

Merkin, J. H. 1976 Free convection boundary layer on an isothermal horizontal cylinder. In *Proc. A.S.M.E.–A.I.Ch.E. Heat Transfer Conf., St. Louis.*

POTTER, J. M. & RILEY, N. 1980 Free convection from a heated sphere at large Grashof number. *J. Fluid Mech.* **100**, 769–783.

SEARS, W. R. & TELIONIS, D. P. 1975 Boundary-layer separation in unsteady flow. *SIAM. J. Appl. Math.* **28**, 215–235.

SIMPSON, C. J. & STEWARTSON, K. 1982*a* A singularity in an unsteady free-convection boundary layer. *Q. J. Mech. Appl. Math.* **35**, 291–304.

SIMPSON, C. J. & STEWARTSON, K. 1982*b* A note on boundary-layer collision on a rotating sphere. In preparation.

STEWARTSON, K. & SIMPSON, C. J. 1982 On a singularity initiating a boundary-layer collision. *Q. J. Mech. Appl. Math.* **35**, 1–16.

STEWARTSON, K., SIMPSON, C. J. & BODONYI, R. J. 1982 The unsteady boundary layer on a rotating disc in a counter-rotating fluid II. In preparation.